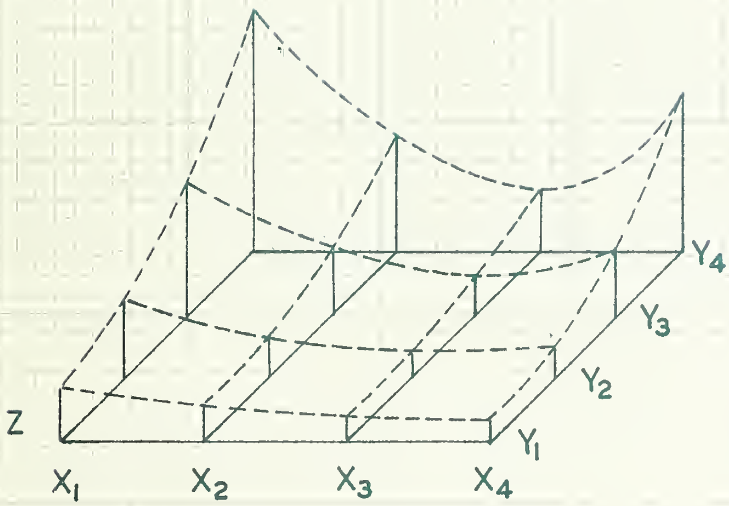
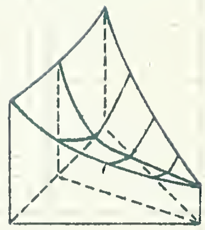
Algebraic Description of Forms in Space

Chester E. Jensen





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R. D. Lane, Director

ALGEBRAIC DESCRIPTION OF FORMS IN SPACE

by

Chester E. Jensen[1/]

The expected spatial relation between continuous variables is often expressed graphically early in a regression or correlation analysis. In turn, the graph may be expressed algebraically. To quantify or statistically evaluate the relation, the algebraic form may be fitted mathematically to a relevant data set. How to approximate the graph algebraically in such a process is the concern here.

Given a dependable basis for the expected relation, curve form bias introduced algebraically and associated information loss will tend to diminish as the algebraic form approaches that of the graph. When the analyst's confidence in the graphed form is great, he should then be prompted to adopt a high standard of descriptive accuracy and needs a fairly exacting descriptive technique.

To achieve this, direct description has some advantages over indirect description. These can be seen in a comparison of the two approaches.

1/ Statistician, Central States Forest Experiment Station, U.S. Department of Agriculture, Forest Service.

Indirect Approach --- that of selecting a form involving a limited array of commonly used transformations of the independent variables ---

$$\text{e.g.} \quad Y = a + b_1X + b_2X^{-1} + b_3X^2 + b_4 \log X$$

--- fitting this form to a relevant data set --- statistically screening components for those most able to reduce the variance about regression --- and, hoping the remaining ones will comprise a form reasonably close to the graphic standard.[2/]

In this case, useful advance knowledge regarding the form of the relation is ignored. The amount of information salvaged from the data alone depends on data strength and on the degree to which the limited set of monomials above can be "molded" into a form like the standard.

Direct Approach --- that of selecting from a large array of transformations and associated coefficients those that jointly are known to duplicate the graphic form with reasonable accuracy[3/] --- fitting the resulting algebraic form to a relevant data set by

---

2/ Where the graphic form is developed from the same data to which it is being fitted by Least Squares, the propriety of statistical evaluation may be questioned.

3/ The standard of accuracy is set by the analyst.

Least Squares --- and, where the graphic form is developed independently of the data to which it is fitted, statistically evaluating the results --- assuming $e \sim NID\ (0, \sigma^2)$.

In this case advance knowledge regarding the form of the relation is as fully utilized as is practicable in establishing the algebraic form --- so that fitting the algebraic form to relevant data will tend to maximize the information extracted.

The iterative procedures presented in this paper represent at least a rudimentary effort to emulate the Direct Approach and to benefit from its attributes.[4/]

It should be noted that neither this nor any other process offers a panacea for curve form description problems. There are likely to be occasions where use of a Direct approach, as demonstrated, will yield substantial improvements over the Indirect Approach --- and others where it will not. Too, success may hinge largely on the ingenuity of the user.

4/ Texts often contain an array of standard algebraic forms from which the analyst may be able to match his "expected" form (1). As in this paper, a more basic understanding of scaling and of algebraic form components is occasionally sought (2).

Examples are given for 16 types of 2-dimensional forms, two 3-dimensional forms, and one 4-dimensional form. Brevity has been sacrificed in order to present a uniform approach to the description problem. But given understanding of the basic approach, you will be in a position to innovate --- the field is virtually limitless.

Relatively high powers of variables are used on occasion to describe sharp curvature, but we are not limited here except by convention and by computational inefficiency.[5/]

Where the form developed involves more than one regression component, be sure that the entire function will be suitably smooth between representative control points from the model.

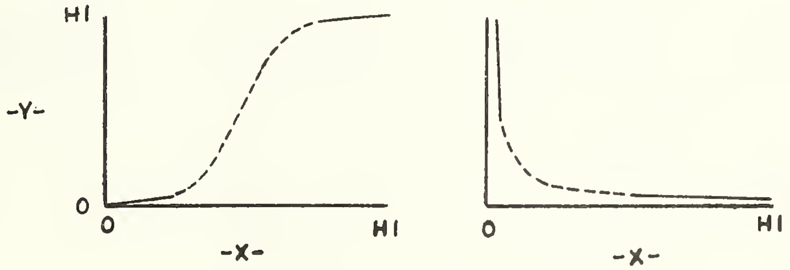
Algebraic forms of the examples have been left unsimplified to facilitate understanding of the source of scaling coefficients. Simplification, if desired, is left to the reader.

The more complex and sometimes more flexible descriptive system involving the ratios of continuous functions should be recognized as extremely important, although not covered here. Examples of "ratio-forms" are given by Elderton (3) in his discussion of Karl Pearson's basic frequency distributions.

5/ An invaluable aid for desk computation is "Barlow's Tables of Squares, Cubes, Square Roots, Cube Roots, etc." or some similar compilation.

THE EXPECTED FORM

It is assumed in this paper that a graphic form has already been established for the relation involved. EXTREME CARE is urged in developing such a form. Before looking at new data, construct an expected response form covering extremes of the independent variable(s). Usually you can be quite certain as to the general slope and value of the curve at such extremes, e.g., flat, tending to zero or vertical, tending to infinity. Then, what happens to the curve between the extremes is rather strongly suggested --- see the dotted lines in the graphed examples below for the shape of the curve between the extremes. Points in the range of the

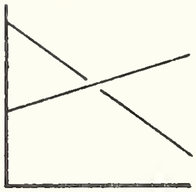


independent variable at which important changes in slope occur are sometimes predictable from past experience. Exert every effort to bring known information to bear on this estimate and quantify the relation where possible.

Then observe the shape of the expected curve in the range of the independent variate that is relevant to the new data. Describe this curve segment algebraically and fit it through the new data by Least Squares. Evaluate statistically as desired.

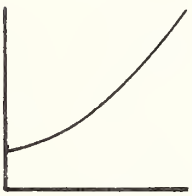
INDEX TO DESCRIPTIVE PROCEDURES

2 DIMENSIONS

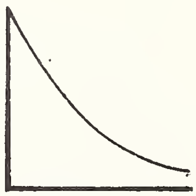


STRAIGHT-LINE CASES--- AN INTRODUCTION TO SCALING

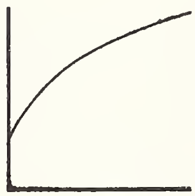
TYPE - 1, P.8



TYPE - 2, P.9



TYPE - 3, P.15



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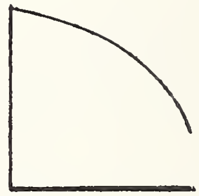
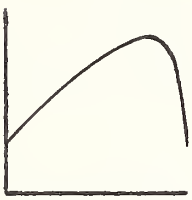
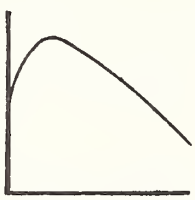


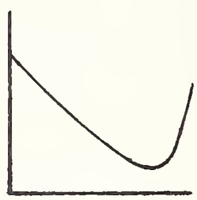
TYPE - 5, P. 17



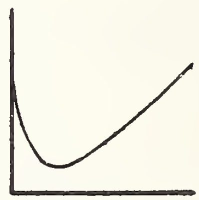
TYPE - 6, P. 18



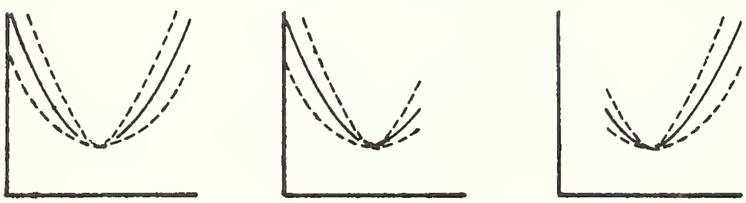
TYPE - 7, P. 21



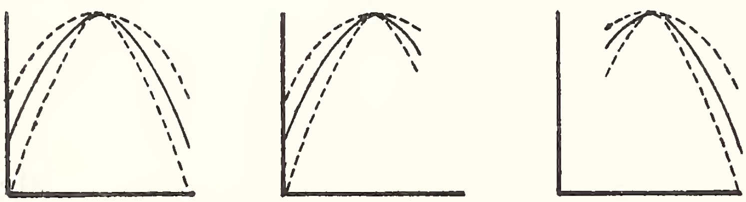
TYPE - 8, P. 22



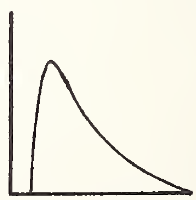
TYPE - 9, P.23



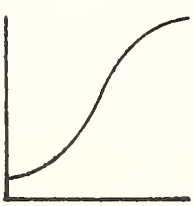
TYPE - 10, P. 24



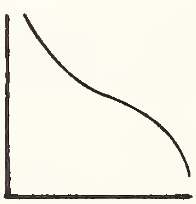
TYPE - 11, P. 26



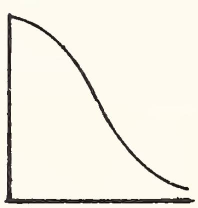
TYPE - 16, P. 32



TYPE - 12, P.27

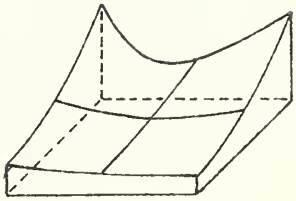


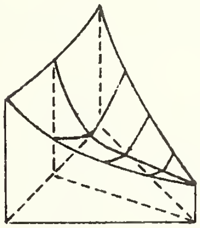
TYPE - 13, P. 30

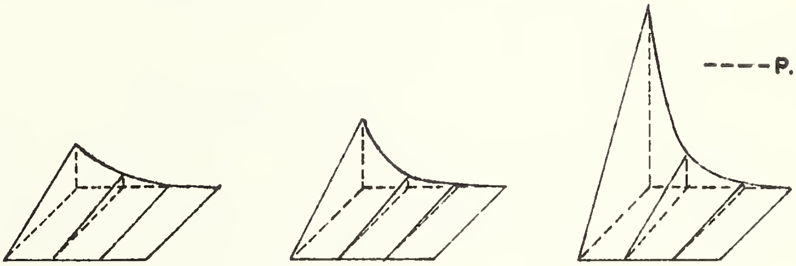


TYPE - 14, P.31

TYPE - 15, P.31







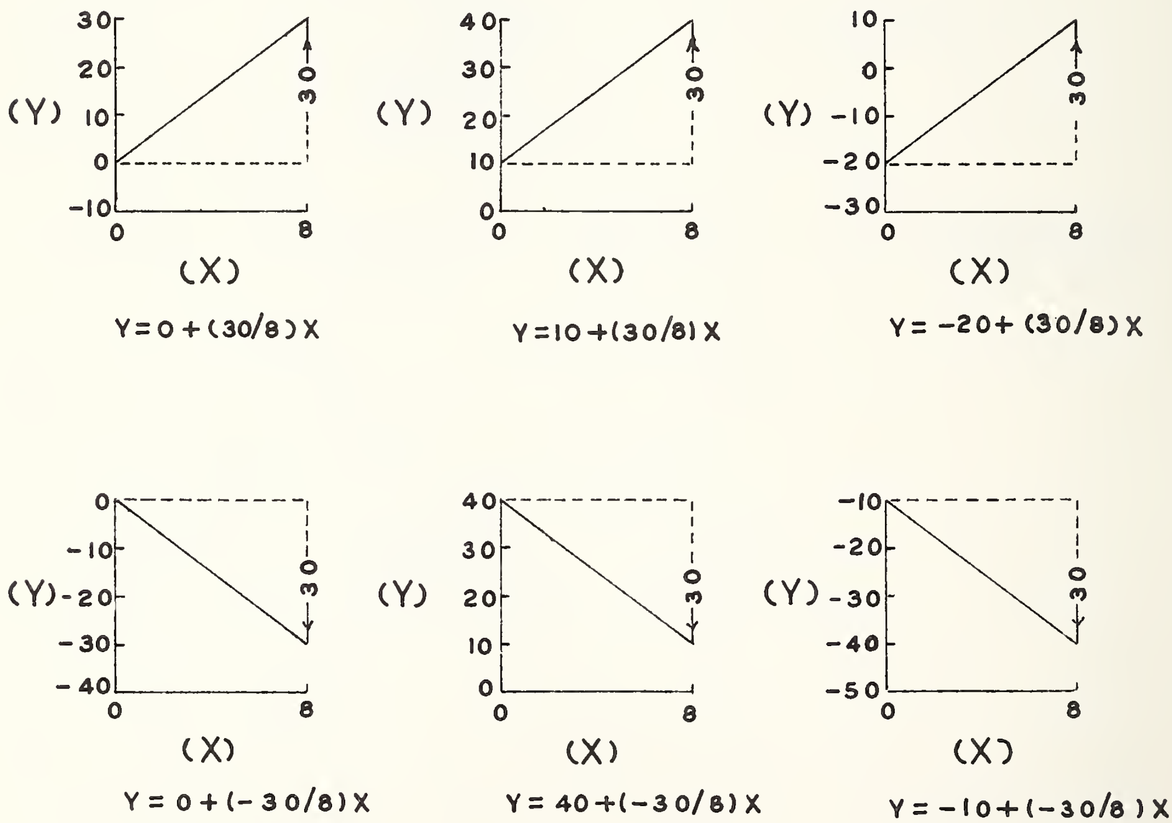
TWO DIMENSIONS

TYPE 1

INTRODUCTION TO SCALING

Examples of positive and negative slopes from various intercepts[6/] are presented below. Note in each case that Y must be described in terms of X, Y-values being obtained by scaling a given value of X to a corresponding value of Y.

$$Y = A + BX$$



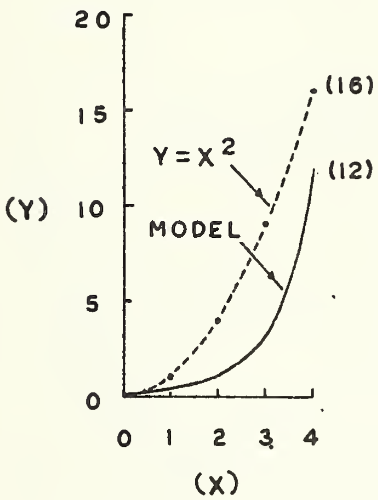
6/ Intercept --- Y-value at X = 0

TYPE 2

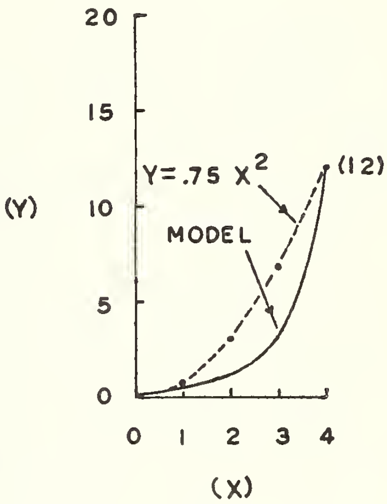
(oriented at Y = 0)

We might start here by seeing how well some scaled powers of X will match the curve or model at representative points (dots in the graphs below). For example, let's try X^2 ---

Before scaling



After scaling



Scale computation

Where $b = Y/X^2$, at the largest value of X.[7/]

$b = (12/16) = .75$

and $Y = a + bX^2$

$Y = 0 + .75\ X^2$

7/ Generally a good working rule. Although sometimes useful, scaling at some other value of X will not be demonstrated until later.

After scaling X^2, we can see that this curve is too flat to represent the model satisfactorily. Because higher powers of X have more curvature, X^4 and X^6 might be tried next.

<u>Scale computations</u>

For X^4, $b = (12/256) = .0469$

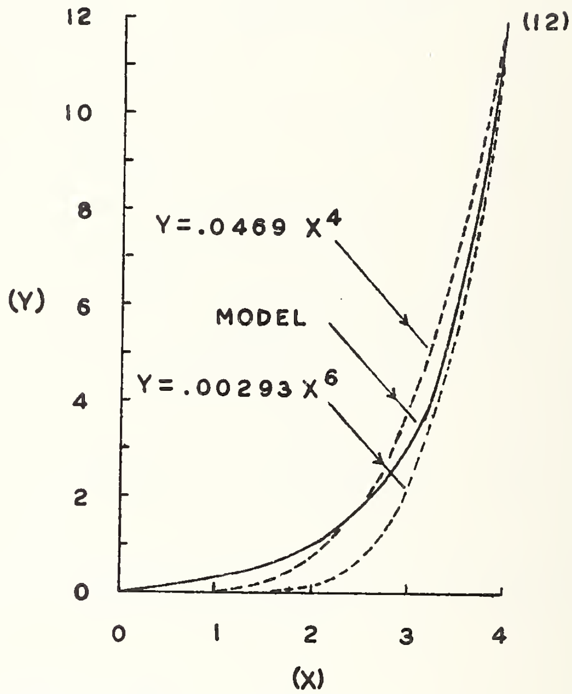
For X^6, $b = (12/4096) = .00293$

And,

$Y = 0 + .0469\ X^4$

$Y = 0 + .00293\ X^6$

<u>After scaling</u>



We might be satisfied with the proximity of $Y = 0 + .0469\ X^4$ to the model.

But, the above example only illustrates the relative curvature of some <u>integral</u> powers of X. On occasion, it may be found that <u>fractional</u> powers of X 8/ are more appropriate.

8/ Fractional powers of X may be computed from X^n = antilog of (n log X).

We can also resort to additive combinations of various powers of X. Suppose we were not satisfied with $Y = 0 + (.0469)\ X^4$, as a description of the original model. We might try to improve that description as follows:

From an examination of the model, it appears that Y-values for X = 0 through X = 1.5 would be fairly well represented by a straight line, Y = a + bX ---

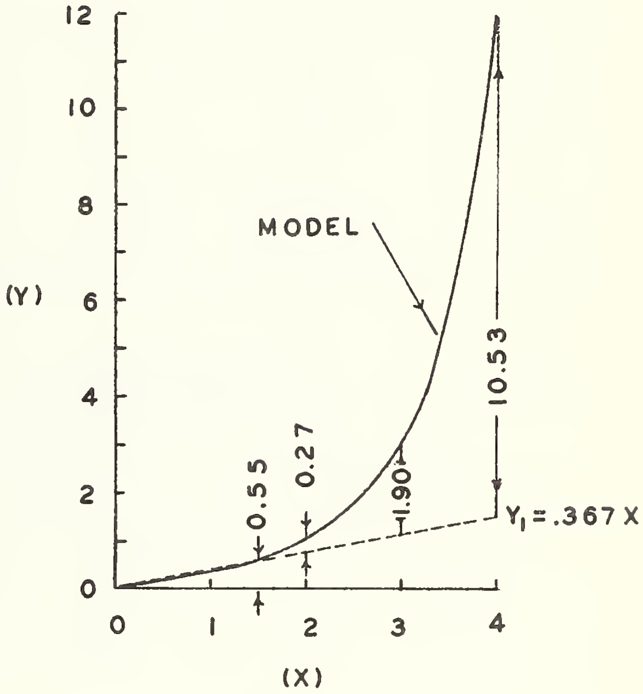
Where,

$b_1 = (Y/X)$ at X = 1.5

$b_1 = (.55/1.5) = .367$

And,

$Y_1 = 0 + .367\ X$



Plotting this form over that of the model, (graph at right) we see that the quantities that must still be added to match the model are of the Type I curve form.

So, let's add a higher power of X as before, but scale it to the largest additional quantity needed, 10.53 here. Also, we must refrain from adding large quantities to the already satisfactory straight line to the left of X = 1.5 --- . Then the power of X we decide to use must be associated with a small "b" to suppress scaled values at X = 1.5 and below.

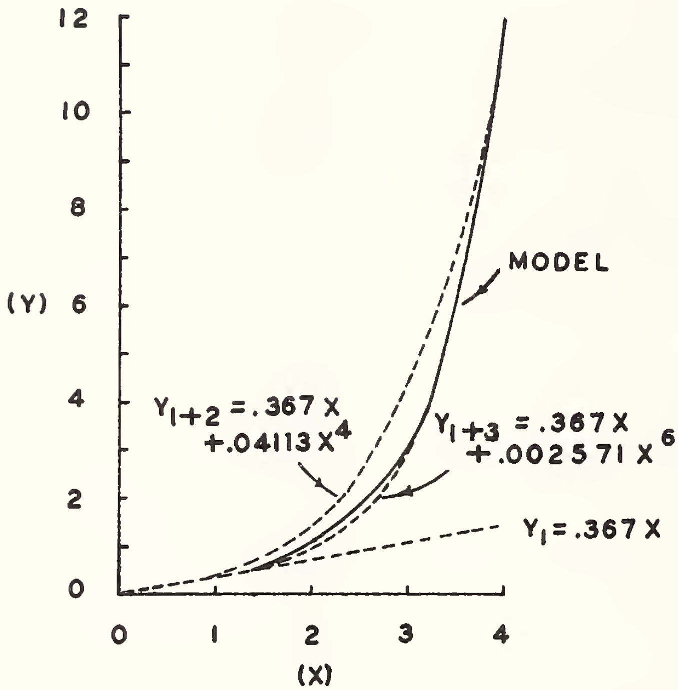
Let's try X^4 and X^6 --- For X^4, $b_2 = (10.53/256) = .04113$

--- For X^6, $b_3 = (10.53/4096) = .002571$

and $Y_2 = 0 + .04113\ X^4$

$Y_3 = 0 + .002571\ X^6$

Adding these forms to the straight line already plotted, we have ---



From the graph above, we can readily see that in matching the model, the better of the two curves is $Y_{1+3} = .367\ X + .002571\ X^6$ --- and that it is, in fact, better than $Y = X^2$, X^4, or X^6 alone (tried previously).

A bookkeeping system follows for the computations ---

Model Values		Y_1	Added amt. to match model		Y_3	Difference from model
X	Y	(.55/1.5) X = .367 X	$Y-Y_1$	X^6	$(10.53/4096)\ X^6 = .002571\ X^6$	$Y-(Y_1 + Y_3)$
0.0	0.00	0.00	0.00	0	0.00*	0.00
1.0	0.30	0.37	-0.07	1	0.00*	-0.07
1.5	0.55	0.55	0.00	11	0.03*	-0.03
2.0	1.00	0.73	0.27	64	0.16	0.11
3.0	3.00	1.10	1.90	729	1.87	0.03
4.0	12.00	1.47	10.53	4096	10.53	0.00

* negligible additions to Y_1 for X = 1.5 or less

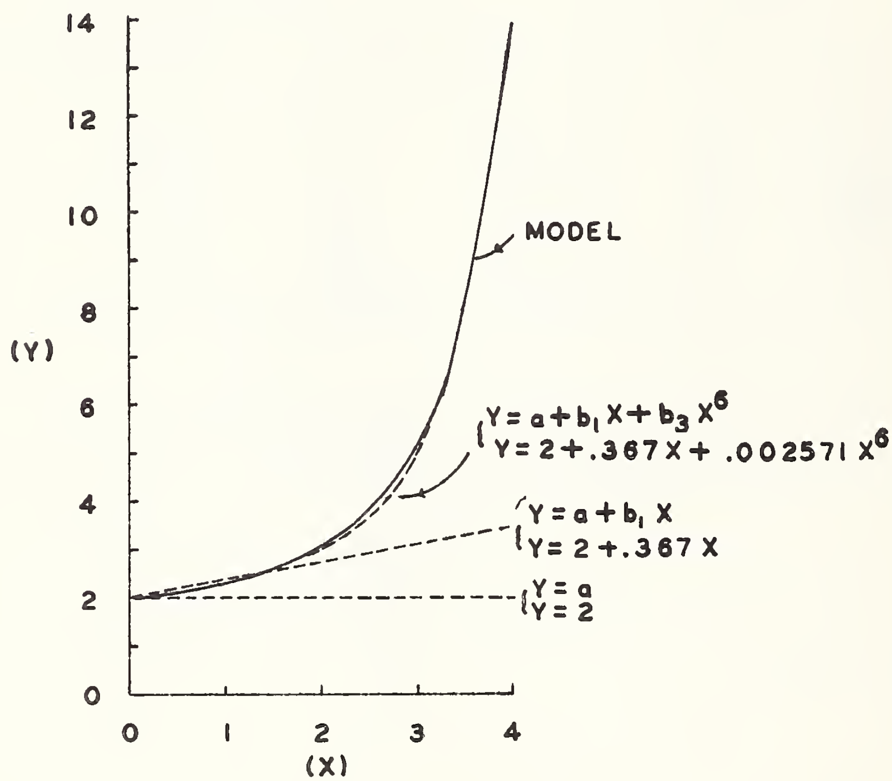
Note that $Y_{1+3} = 0.367\ X + 0.002571\ X^6$ differs from the model by 0.11 or less for the X-values examined over the range of X. Our final curve form here is $Y = a + b_1X + b_3X^6$. A more exact description of the model might or might not be found by trying other forms.[9/]

9/ In place of 0.367 X, we might start with a fairly flat power curve, such as, $b_4X^{1.6}$ --- perhaps next adding b_5X^6 --- etc.

TYPE 2

(not oriented at X,Y = 0)

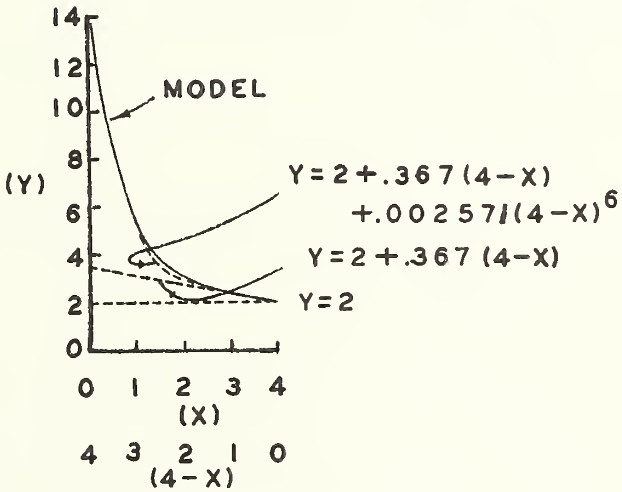
Where the model intersects the Y-axis at some point other than zero, the curve development procedure would be identical to that of the foregoing case, but straight-line and curved forms would now be added to the intercept value, Y = a.



As before, final form adopted here is $Y = a + b_1X + b_3X^6$.

TYPE 3

The procedure here is basically the same as that outlined for Type 2. Note the transformation of X, (4-X), for the last two terms --- to make additions to regression larger as X approaches zero. About the same effect could be obtained using $1/X^n$, $X > 0$, $n \geq 1$, but this transformation is a little more difficult to compute. Also, starting the building process at a = 14 would permit use of the untransformed "X" in $Y = 14+(-b_1)X+b_2X^n$ --- but text continuity is best served here and subsequently by adopting (4-X) when appropriate --- either transform is fully acceptable.

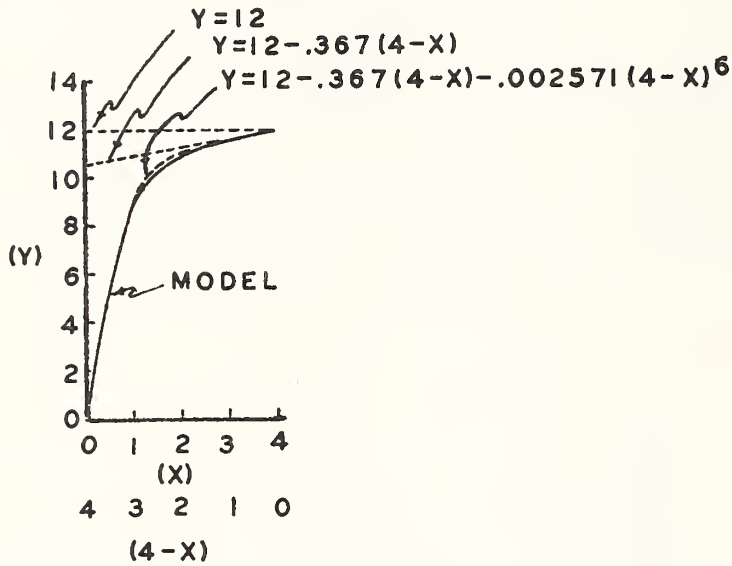


The final form description adopted in the above example is

$Y = a + b_1(4\text{-}X) + b_2(4\text{-}X)^6$ --- $0 \leq X \leq 4$

TYPE 4

The procedure here is basically the same as that outlined for TYPE 2 except that we are working from high to low values of Y. Note the transformation of X, (4-X), for the last two terms to make reductions from the constant (12) larger as X approaches zero.

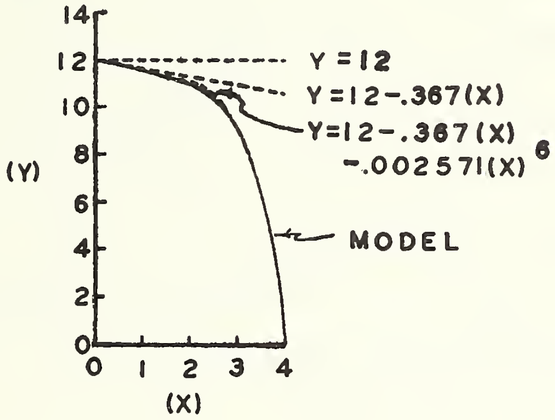


The final curve form description adopted in the above example is $Y = a + b_1(4\text{-}X) + b_2(4\text{-}X)^6,\ 0 \leq X \leq 4$

Another possibility here is to start with Y = 0, adding scaled, positive fractional powers of X --- $(X^{1/n})$ --- to arrive at a form like that of the model. But once again, $X^{1/n}$ is more difficult to compute.

TYPE 5

The procedure here is basically the same as that outlined for TYPE 2, except that we are working from high to low values of Y.

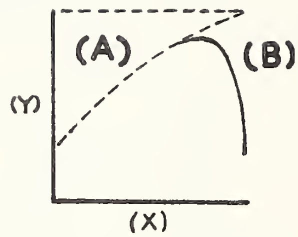


The final curve form description adopted in the above example is $Y = a + b_1X + b_2X^6$

TYPE 6

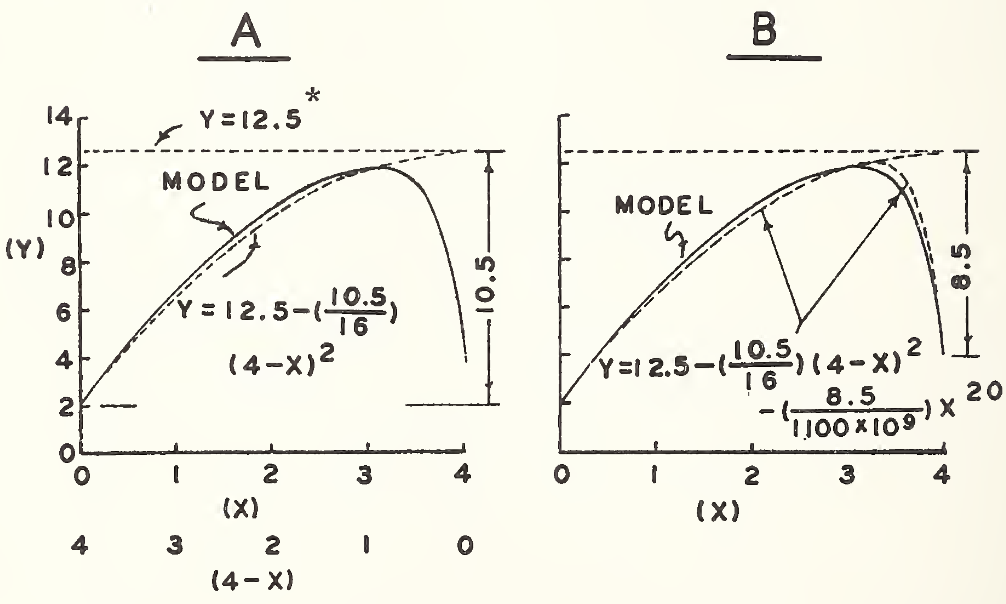
Consider this form

in two pieces.



1. Describe part (A) as per TYPE 4.
2. Subtract a high power of X from the (A) form to give the sharp downward trend of part (B) ---

For example, using the model below:



* From extension of the model trend.

To match part (A) of the original model, we transform X to $(4-X)^2$ and scale this quantity to 10.5 --- (12.5 - 2.0) at the largest value of (4-X). We find that $Y = a - b(4-X)^2$ is fairly suitable. Scaled, it is $Y = 12.5 + (-10.5/16)\ (4-X)^2$.

To match part (B) of the model, we must find some high power of X that, when scaled to 8.5 and subtracted from our description for part (A), will subtract little or nothing from X=0 through X=3.

Try X^{20}, then $b = (-8.5/1100 \times 10^9) = -0.773\ (10^{-11})$ and the complete model can be described as ---

$$Y = 12.5 - 0.656(4-X)^2 - 0.773(10^{-11})X^{20} \text{ --- } 0 \leq X \leq 4$$

Summary of Computations:

Part (A)

Model		Y_1				Y_2	
X	Y	a*	$Y - Y_1$	(4-X)	$(4-X)^2$	$\frac{-10.5}{16}(4-X)^2$	$Y-(Y_1+Y_2)$
0	2.0	12.5	- 10.5	4	16	- 10.50	0
1	6.8	12.5	- 5.7	3	9	- 5.90	.20
2	10.2	12.5	- 2.3	2	4	- 2.62	.32
3	11.8	12.5	- .7	1	1	- .66	- .04
4	4.0	12.5	- 8.5	0	0	- 0	- 8.50

Part (B)

Model			Y_3	diff. from model
X	Y	X^{20}	$\frac{-8.5}{1,100MMM} X^{20}$	$Y-(Y_1+Y_2+Y_3)$
0	2.0	0	0	0
1	6.8	1	0	.20
2	10.2	$1,049\ (10^3)$	0	.32
3	11.8	$3,487\ (10^6)$	- .03	- .01
4	4.0	$1,100\ (10^9)$	- 8.50	0

* From extension of model trend

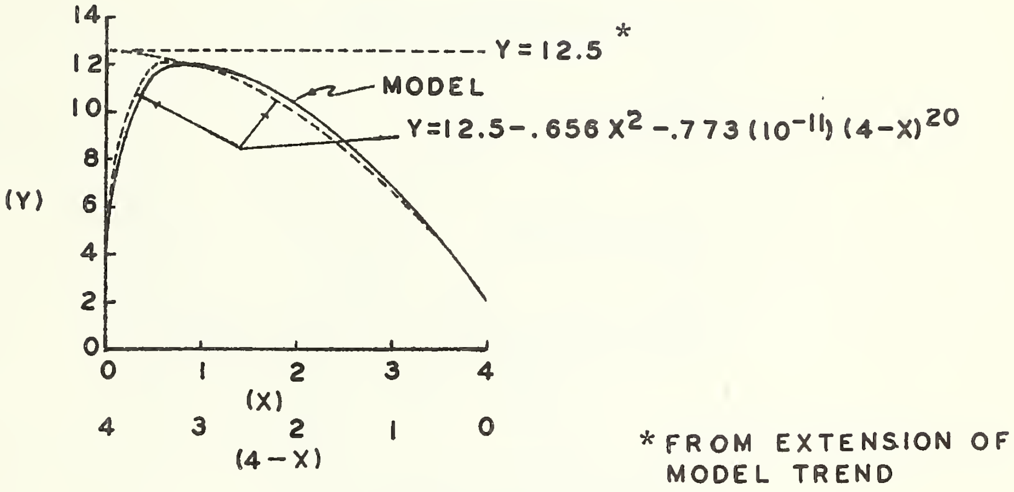
Providing the above "difference from model" is not considered excessive, the accepted final form would then be ---

$$Y = a + b_1(4-X)^2 + b_2(X)^{20}$$

$$--- \quad 0 \leq X \leq 4$$

TYPE 7

The procedure outlined for TYPE 6 is almost identical to the one applied here --- follow the steps in the graph below.

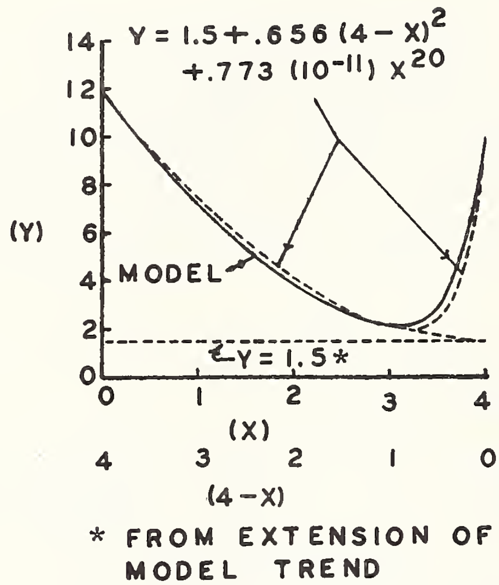


Then, the final form here is ---

$$Y = a + b_1X^2 + b_2(4\text{-}X)^{20} \text{ --- } 0 \leq X \leq 4$$

TYPE 8

The procedure outlined for TYPE 6 is very similar to the one applied here --- follow the steps in the graph below.

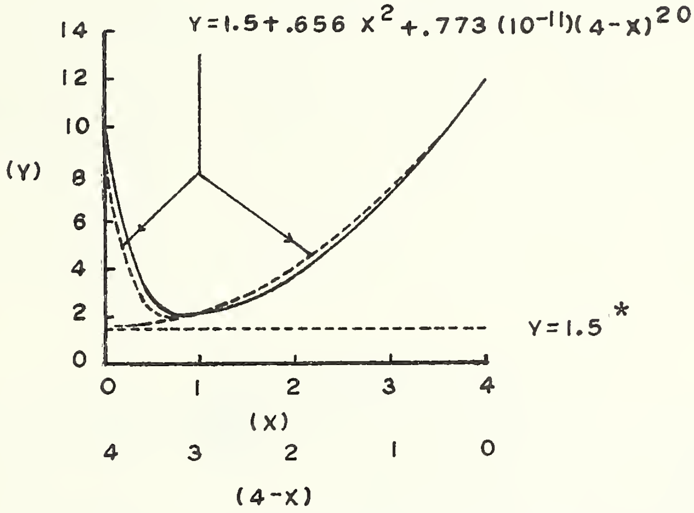


The final form is then ---

$$Y = a + b_1(4-X)^2 + b_2X^{20} \quad --- \quad 0 \leq X \leq 4$$

TYPE 9

The procedure outlined for TYPE 6 is very similar to the one applied here --- follow the steps in the graph below.



* FROM EXTENSION OF MODEL TREND

The final form adopted is then ---

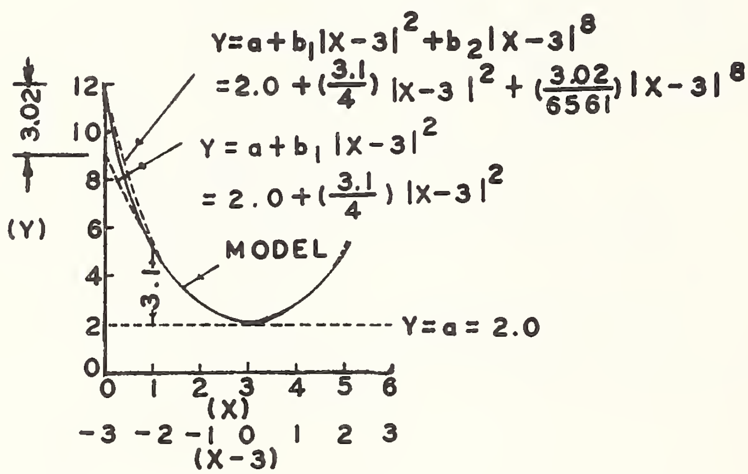
$$Y = a + b_1X^2 + b_2(4-X)^{20} \quad \text{---} \quad 0 \leq X \leq 4$$

TYPE 10

Where the curvature on either side of the minimum is more or less symmetrical, apply TYPE 2 processes to the transformation of X given below.

Determine "X" at the minimum value of "Y" and subtract from all X-values. Then, using the absolute values[10/] of these differences, proceed to locate descriptive terms for either side of the model.

For example:



10/ Use of the absolute value permits adoption of any integral or fractional power of the transform through elimination of negative algebraic signs.

Summary of Computations:

Model		Y_1				Y_2
X	Y	a*	$(Y-Y_1)$	$(X-3)$	$\|X-3\|^2$	$\frac{(3.1)}{4}\|X-3\|^2$
0	12.0	2	10.0	-3	9	6.98
1	5.1	2	3.1	-2	4	3.10
2	2.8	2	.8	-1	1	.78
3	2.0	2	.0	0	0	.00
4	2.8	2	.8	1	1	.78
5	5.1	2	3.1	2	4	3.10

* The minimal point of the graphed curve

Model			Y_3	diff. from model
X	$Y-(Y_1+Y_2)$	$\|X-3\|^8$	$\frac{3.02}{6561}\|X-3\|^8$	$Y-(Y_1+Y_2+Y_3)$
0	3.02	6561	3.02	0.00
1	.00	256	.12	-.12
2	.02	1	.00	.02
3	.00	0	.00	.00
4	.02	1	.00	.02
5	.00	256	.12	-.12

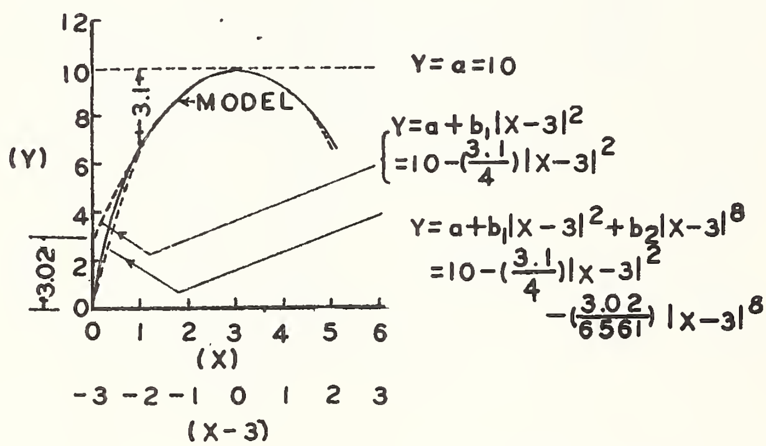
The final form adopted is

$$Y = a + b_1|X-3|^2 + b_2|X-3|^8 \quad --- \quad 0 \leq X \leq 6$$

TYPE 11

Where the curvature on either side of the maximum is more or less symmetrical, apply TYPE 2 processes to the transformation of X given below.

Determine "X" at the maximum value of "Y" and subtract from all X-values. Then, using the absolute values of the differences, proceed to locate descriptive terms for either side of the model as shown for TYPE 2. Note that we are working down from the intercept, Y = 10, here. The computations are directly analagous to those for TYPE 10.



The final form adopted is

$$Y = a + b_1 \; |X-3|^2 + b_2 \; |X-3|^8 \quad --- \quad 0 \leq X \leq 6$$

TYPE 12

When trends on opposite sides of the inflection point, I[11/], are reasonably symmetric in reverse directions, the following procedure may lead to a fairly good description of the model.

1. Let a straight line represent as much of the central section of the model as seems appropriate. Find graphically the Y-value of this line at X = 0 and also the slope of the line. See graph (A) in the example that follows.

2. It will be seen from graph (A) that it is necessary to add a curvilinear quantity to the lower end of the line and to subtract a similar amount from the upper end of the line to match the model.

Transforming "X" to X - (X at the inflection point), we have a series of numbers that are equal but opposite in sign on each side of the inflection point X_I. The sign differences are maintained whenever the integral power of $(X-X_I)$ is odd ---

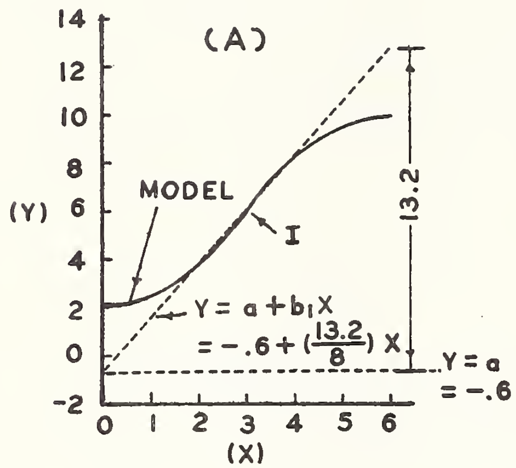
e.g.: $(X-X_I)^1$, $(X-X_I)^3$, $(X-X_I)^5$

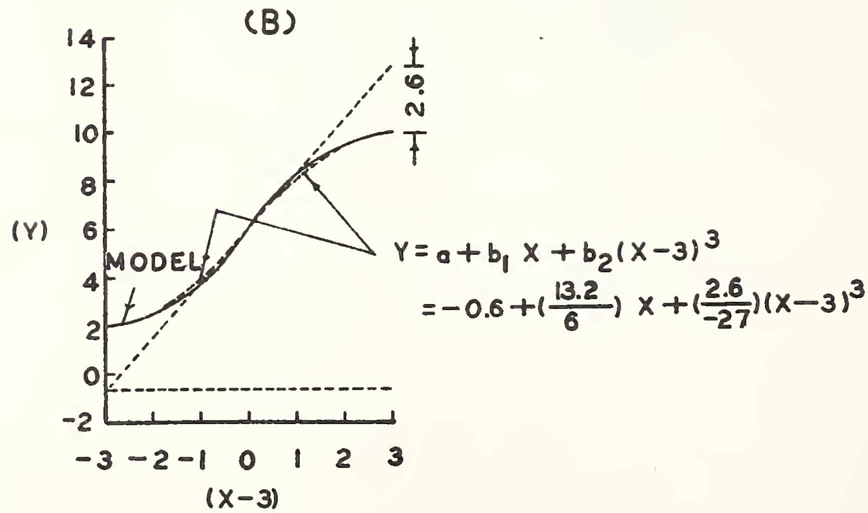
or when the denominators of fractional powers are odd integrals (4) ---

e.g.: $(X-X_I)^{4/3}$, $(X-X_I)^{12/5}$, $(X-X_I)^{4/7}$

11/ Note inflection point, I, in graph (A) which follows.

Then we can use the above powers of $(X-X_I)$ to add and subtract from opposite ends of the straight line, as indicated in graph (B) below. Note that the scaling coefficient for $(X-X_I)^n$ must be small enough to suppress additions to the straight line values that already represent the model with suitable accuracy.





Summary of Computations:

Model		Y_1	Y_2				Y_3	diff.fr.model
X	Y	a*	$\frac{13.2^*}{6}X$	$Y-(Y_1+Y_2)$	$(X-3)$	$(X-3)^3$	$\frac{(2.6)(X-3)^3}{-27}$	$Y-(Y_1+Y_2+Y_3)$
0	2.0	-0.6	0.0	2.60	- 3	-27	2.60	0.00
1	2.4	- .6	2.2	.80	- 2	- 8	.77	.03
2	3.8	- .6	4.4	.00	- 1	- 1	.10	- .10
3	6.0	- .6	6.6	.00	0	0	.00	.00
4	8.2	- .6	8.8	.00	1	1	- .10	.10
5	9.6	- .6	11.0	- .80	2	8	- .77	- .03
6	10.0	- .6	13.2	-2.60	3	27	-2.60	.00

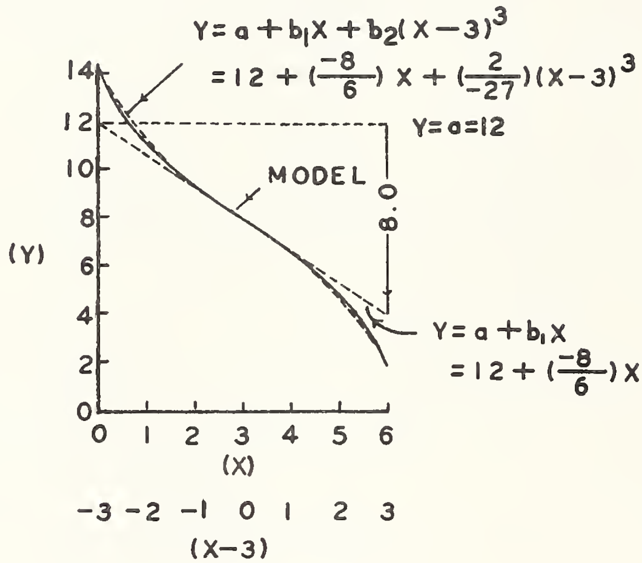
* determined graphically

The final form adopted is

$$Y = a + b_1X + b_2(X-3)^3 \text{ --- } 0 \leq X \leq 6$$

TYPE 13

The procedures for TYPE 12 are directly applicable here, excepting that the sign of the straight line becomes negative. For example:



Summary of Computations:

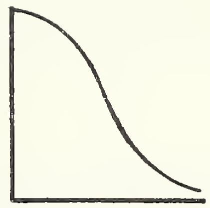
Model		Y_1	Y_2				Y_3	
X	Y	a*	$\left[\frac{-8}{6}\right]^* X$	$Y-(Y_1+Y_2)$	(X-3)	$(X-3)^3$	$\frac{2}{-27}(X-3)^3$	$Y-(Y_1+Y_2+Y_3)$
0	14.0	12	0.00	2.00	-3	-27	2.00	0.00
1	11.1	12	-1.33	.43	-2	- 8	.59	-.16
2	9.3	12	-2.67	-.03	-1	- 1	.07	-.10
3	8.0	12	-4.00	.00	0	0	.00	.00
4	6.7	12	-5.33	.03	1	1	- .07	.10
5	4.9	12	-6.67	-.43	2	8	- .59	.16
6	2.0	12	-8.00	-2.00	3	27	-2.00	.00

* determined graphically

The final form adopted is

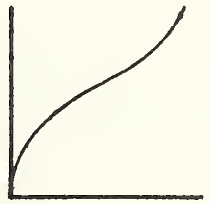
$$Y = a + b_1X + b_2(X-3)^3 \quad --- \quad 0 \leq X \leq 6$$

TYPE 14



The procedures for TYPE 13 are directly applicable here excepting that the sign of b_2 in $b_2(X-X_I)^n$ is positive.

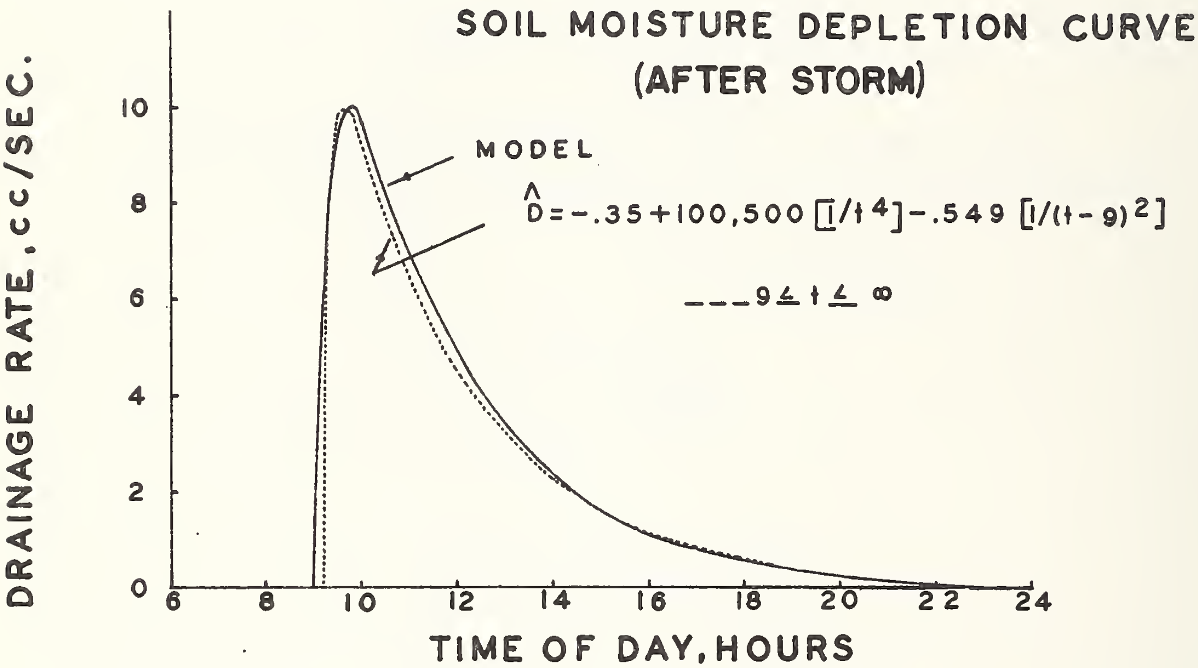
TYPE 15



The procedures for TYPE 12 are directly applicable here excepting that the sign of b_2 in $b_2(X-X_I)^n$ is positive.

TYPE 16

The model below is a mixture of TYPES 3 and 7 (pages 14 and 20 respectively) applied to a particular research generated graphic form.



A tabular summary of the algebraic form development follows.

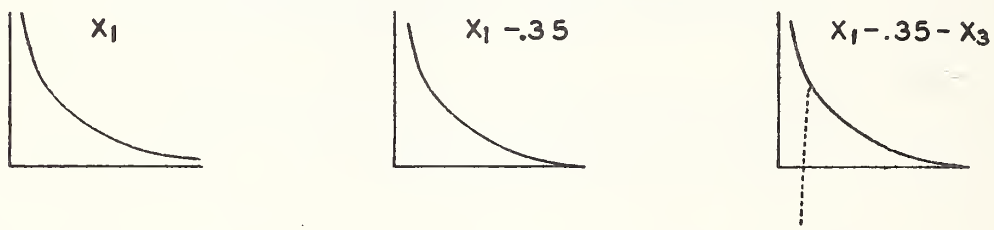
SUMMARY OF COMPUTATIONS

MODEL				X_1	X_2			A	X_3	$\hat{D}$
Time of day, hours t	Drainage rate, cc/sec. D	t^4	$(x10^{-4})$ $1/t^4$	$(x10^4)$ $10.05(1/t^4)$	X -.35	t-9	1/(t-9)	$1/(t-9)^2$	-.549A	$X_2 + X_3$
9.00	0	6,561	1.524	15.32	14.97	0	∞	∞	$-\infty$	$-\infty$
9.17	4.4	7,071	1.414	14.21	13.86	.167	5.988	35.86	-19.69	-5.83
9.33	7.9	7,578	1.320	13.27	12.92	.333	3.003	9.02	- 4.95	7.97
9.50	9.4	8,145	1.228	12.34	11.99	.500	2.000	4.00	- 2.20	9.79
9.67	9.9	8,744	1.144	11.50	11.15	.667	1.499	2.25	- 1.24	9.91
9.83	10.0	9,337	1.071	10.76	10.41	.833	1.200	1.44	- .79	9.62
10.00	9.7	10,000	1.000	10.05	9.70	1.000	1.000	1.00	- .55	9.15
10.17	9.2	10,698	.935	9.40	9.05	1.167	.857	.73	- .40	8.65
10.33	8.7	11,387	.878	8.82	8.47	1.333	.750	.56	- .31	8.16
10.50	8.2	12,155	.823	8.26	7.91	1.500	.667	.44	- .24	7.67
11.00	6.9	14,641	.683	6.86	6.51	2.000	.500	.25	- .14	6.37
12.00	4.9	20,736	.482	4.84	4.49	3.000	.333	.11	- .06	4.43
13.00	3.5	28,561	.350	3.52	3.17	4.000	.250	.06	- .03	3.14
14.00	2.4	38,416	.260	2.61	2.26	5.000	.200	.04	- .02	2.26
15.00	1.6	50,625	.198	1.99	1.64	6.000	.167	.03	- .02	1.64
16.00	1.1	65,536	.153	1.54	1.19	7.000	.143	.02	- .01	1.19
17.00	.8	83,521	.120	1.21	.86	8.000	.125	.02	- .01	.86
18.00	.6	104,976	.095	.95	.60	9.000	.111	.01	- .01	.60
20.00	.3	160,000	.062	.62	.28	11.000	.091	.01	- .01	.28
23.00	0	279,841	.036	.36	.01	14.000	.071	.00	.00	.01

NOTES

1. Here, reciprocals of the independent variable, t, replace powers of the linear transposition --- (4-X) --- shown for TYPES 3 and 7. The reciprocals happened to be more accurate in this case.

2. X_1 describes the back-slope of the model and is scaled to D +.35 at t = 10.00 --- scaling coefficient = $(9.70 +.35)/1.000 \text{ X } 10^{-4}$ = $10.05 \text{ x} 10^4$. The .35 was added at this point because a trial scaling using D alone showed the right tail to be about .35 cc/sec. high.



3. In X_2 --- (or, X_1-.35) --- the -.35 corrects for the high right tail of the back-slope.

4. And last, appropriately increasing amounts, $-.549/(t-9)^2$, were subtracted from X_2 as t became smaller. "9" was used as the point of origin for the t-transform, $(t-9)^2$, because 9 was the origin of the model. The scaling coefficient for X_3 in this case was based on the difference between X_2 and D of the model at <u>t = 9.67</u> --- X_2 = 11.15, D = 9.90, X_2-D = 1.25 --- $1/(t-9)^2$ = 2.25 --- coeff. = 1.25/2.25 = .551 --- finally adjusted to .549 for slight benefit in the region to the left of maximum D.

THREE DIMENSIONS

GRAPHIC DEVELOPMENT OF SURFACES

Before discussing the mathematical description of multi-dimensional models, it will probably be desirable to consider the graphic development of surfaces from sets of data. A fairly straight-forward procedure is presented in the Appendix, beginning on page 1.

ALGEBRAIC DESCRIPTION

The analyst's objectives and the degree of his confidence in the adopted model will probably control the extent of his algebraic description effort. Just as for 2-dimensional models, we will consider the case where a fairly accurate description of a well-established model is desired and where complexity of algebraic description is not a deterrent in either fitting or use of the model.

Example #1. It will be seen here that the algebraic form derived is uncommonly complex, involving five transformations of the two independent variables, which is the price, in this case, for a reasonable facsimile of the model. Given additional effort, it is possible that a simpler, more accurate form could be developed. But the example is retained because it is fully applicable "as is" and demonstrates the descriptive process well.

The solid-line figure in the following graph is the model to be described.

LEGEND

—— MODEL SURFACE

- - - - - $\hat{Z} = .51 + .00896(7-X)^{2.5} + [.02432(X)^{1.85}(\frac{Y}{15})] + [.6384(10^{-8})(7-X)^{10}(\frac{Y}{15})]$

$+ [.0007764(X)^{3.6}][.027988(\frac{Y}{15})^{7}] + [.8534(10^{-8})(7-X)^{10}][.027988(\frac{Y}{15})^{7}]$

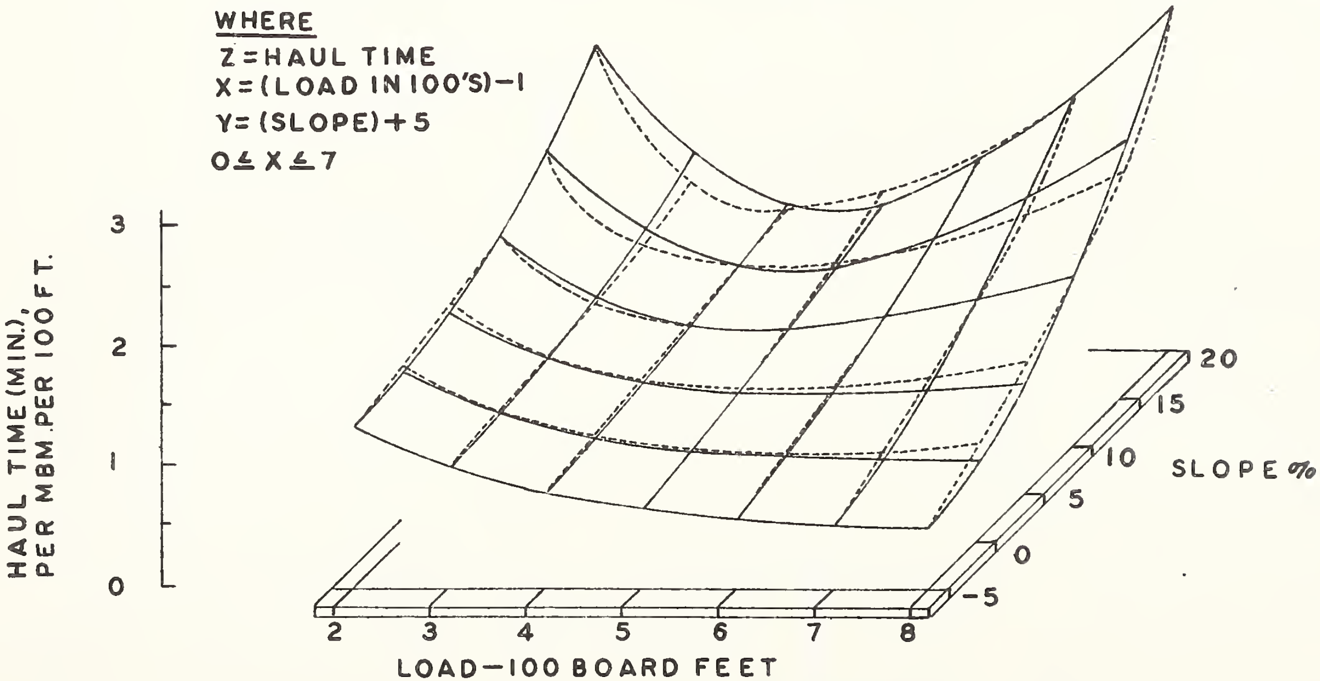
WHERE

Z = HAUL TIME

X = (LOAD IN 100'S) − 1

Y = (SLOPE) + 5

$0 \leq X \leq 7$



MODEL SURFACE VS. A DERIVED MATHEMATICAL DESCRIPTION OF THAT SURFACE

To describe the model algebraically, start by selecting enough surface values over the ranges of both independent variates (X,Y) to specify major surface trends adequately. But keep the number of such values to a minimum --- Z_i for twelve (X,Y)-combinations were adopted here.

Then proceed to find form components that jointly describe an acceptably smooth surface passing through the Z-points adopted. The final components are shown below in tabular form in preference to lengthy verbal description. Note how successive components describe additional portions of the model much as they did for 2-dimensional cases. Components parallel those for curve TYPES 3 and 9. Other surfaces may involve any or all of TYPES 1 - 16.

Two added features: scaled transformations of the Y-values are used to suppress additions to portions of the surface for which satisfactory X-descriptive terms have already been located[12/] and (X,Y)-cross products are developed to describe joint changes in X and Y.

Each operation must be carefully thought through in order to follow the steps outlined. The exponent adopted for any term is the best of perhaps five or six tried. The entire description here required two days for development.

A graphic portrayal of the regression components follows the derivation tables.

12/ Both X and Y are independent variates here.

DEVELOPMENT OF ALGEBRAIC DESCRIPTION OF SURFACE MODEL

Model Y	Model X	Model Z	Z_1 1/	$(Z-Z_1)$	$(7-X)$	$(7-X)^{2.5}$	Z_2: $\left[\frac{.79}{88.2}\right](7-X)^{2.5}$
0	1	1.30	0.51	0.79 2/	6	88.20	0.790
	3	.80	.51	.29 2/	4	32.00	.287
	5	.58	.51	.07 2/	2	5.66	.051
	7	.51	.51	.00 2/	0	.00	.000
15	1	1.71	.51	1.20	6	88.20	.790
	3	.98	.51	.47	4	32.00	.287
	5	1.04	.51	.53	2	5.66	.051
	7	1.40	.51	.89	0	.00	.000
25	1	2.50	.51	1.99	6	88.20	.790
	3	1.20	.51	.69	4	32.00	.287
	5	1.61	.51	1.10	2	5.66	.051
	7	2.85	.51	2.34	0	.00	.000

Model Y	Model X	$Z-(Z_{1+2})$	$X^{1.85}$	A: $\left[\frac{.89}{36.6}\right]X^{1.85}$	$\frac{Y}{15}$	Z_3: $A\left[\frac{Y}{15}\right]$	$Z-(Z_{1+2+3})$	$(7-X)^{10}$	Z_4: $\left[\frac{.386}{60,466M}\right](7-X)^{10}\left[\frac{Y}{15}\right]$
0	1	0.000	1.00	0.024	0	0	0.000	60,466M	0
	3	.003	7.63	.186	0	0	.003	1,049M	0
	5	.019	19.64	.478	0	0	.019	1,024	0
	7	.000	36.60	.890	0	0	0	0	0
15	1	.410 3/	1.00	.024	1	0.024	.386	60,466M	.386
	3	.183 3/	7.63	.186	1	.186	- .003	1,049M	.007
	5	.479 3/	19.64	.478	1	.478	.001	1,024	0
	7	.890 3/	36.60	.890	1	.890	0	0	0
25	1	1.200	1.00	.024	1.667	.040	1.160	60,466M	.643
	3	.403	7.63	.186	1.667	.310	.093	1,049M	.011
	5	1.049	19.64	.478	1.667	.797	.252	1,024	0
	7	2.340	36.60	.890	1.667	1.484	.856	0	0

1/ A constant --- the lowest value for the surface.
2/ Curve Type 3 --- well described by Z_2.
3/ Curve Type 9 --- described by $Z_3 + Z_4$ --- note that Y is used to suppress additions to $Z_1 + Z_2$, already satisfactory at Y = 0.

Y	X			B		C	Z_5
		$Z-(Z_{1+\ldots+4})$	$X^{3.6}$	$\left[\frac{.856}{1102.5}\right] X^{3.6}$	$\left[\frac{Y}{15}\right]^7$	$\left[\frac{1}{35.78}\right]\left[\frac{Y}{15}\right]^7$	BC
0	1	0.000	1.0	0.001	0	0	0
	3	.003	52.2	.041	0	0	0
	5	.019	328.3	.255	0	0	0
	7	.000	1102.5	.856	0	0	0
15	1	.000	1.0	.001	1	.028	0
	3	-.010	52.2	.041	1	.028	.001
	5	.001	328.3	.255	1	.028	.007
	7	.000	1102.5	.856	1	.028	.024
25	1	.517	1.0	.001	35.78	1	.001
	3	.082	52.2	.041	35.78	1	.041
	5	.252 1/	328.3	.255	35.78	1	.255
	7	.856	1102.5	.856	35.78	1	.856

Y	X			Z_6	2/
		$Z-(Z_{1+\ldots+5})$	$(7-X)^{10}$	$\left[\frac{.516}{60,466M}\right](7-X)^{10}C$	$Z-(Z_{1+\ldots+6})$
0	1	0.000	60,466M	0	0.000
	3	.003	1,049M	0	.003
	5	.019	1,024	0	.019
	7	.000	0	0	.000
15	1	.000	60,466M	.014	-.014
	3	-.011	1,049M	0	-.011
	5	-.006	1,024	0	-.006
	7	-.024	0	0	-.024
25	1	.516	60,466M	.516	.000
	3	.041	1,049M	.009	.032
	5	-.003	1,024	0	-.003
	7	.000	0	0	.000

1/ Curve Type 9 --- described by $Z_5 + Z_6$ --- note that $\left[\frac{Y}{15}\right]^7$ is used to suppress additions to $Z_1 + Z_2 + Z_3 + Z_4$, already satisfactory at Y = 0 and Y = 15.

2/ These are the aggregate differences between the model values and the mathematical description --- note the small differences in comparison to the original, corresponding model Z values.

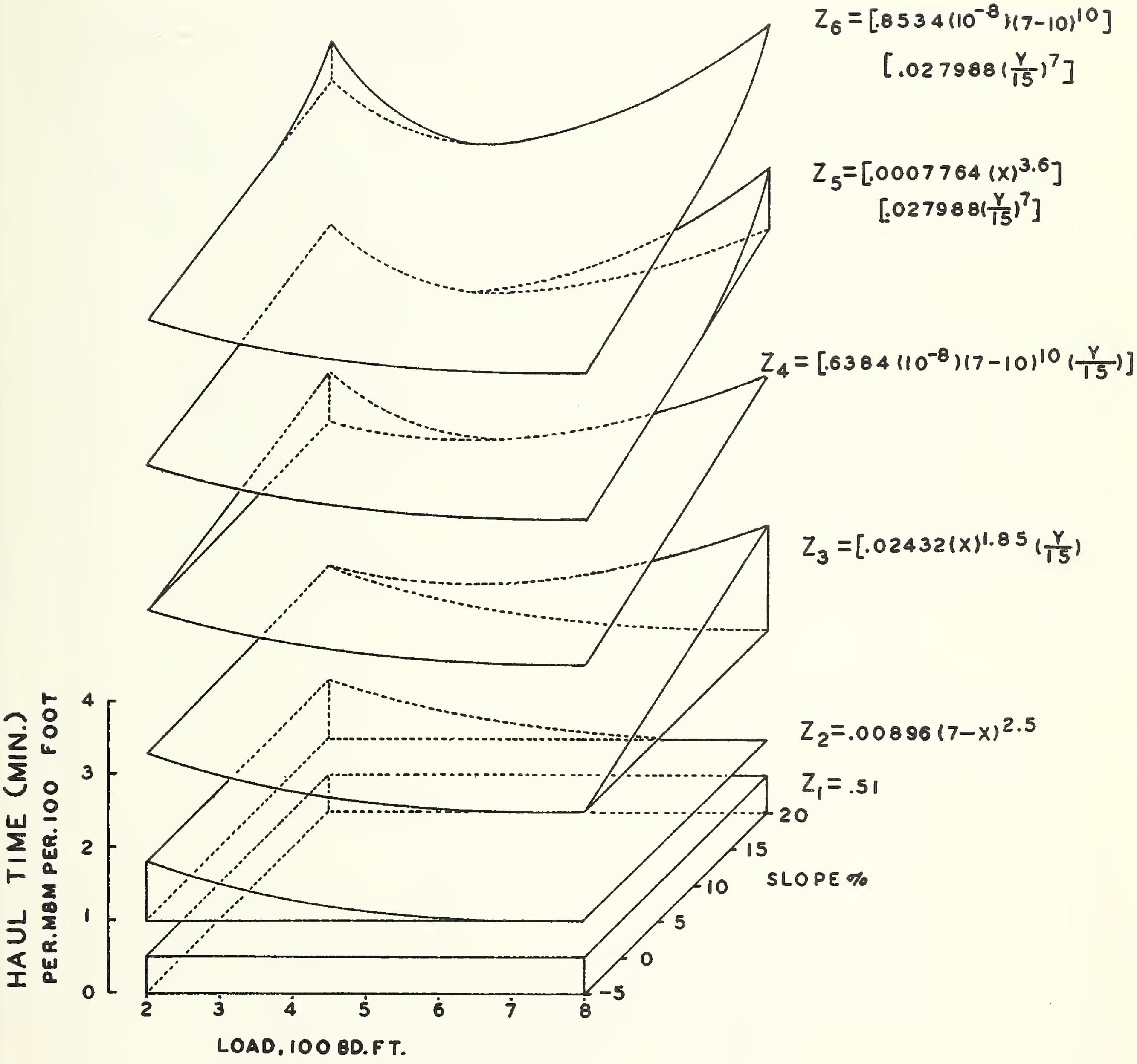
FINAL ALGEBRAIC DESCRIPTION

$$Z = .51 + .00896(7-X)^{2.5} + .02432(X)^{1.85}\left[\frac{Y}{15}\right] + .6384(10^{-8})(7-X)^{10}\left[\frac{Y}{15}\right] + \left[.0007764(X)^{3.6}\right]\left[.027988\left\{\frac{Y}{15}\right\}^{7}\right] + \left[.8534(10^{-8})(7-X)^{10}\right]\left[.027988\left\{\frac{Y}{15}\right\}^{7}\right] \text{---} 0 \leq X \leq 7$$

This form is graphed over that of the model in dotted lines for visual comparison---see the foregoing graph.

Also, see the following graph for contribution to regression by components of the form above.

CONTRIBUTION TO REGRESSION
BY COMPONENTS OF
THE DERIVED FORM



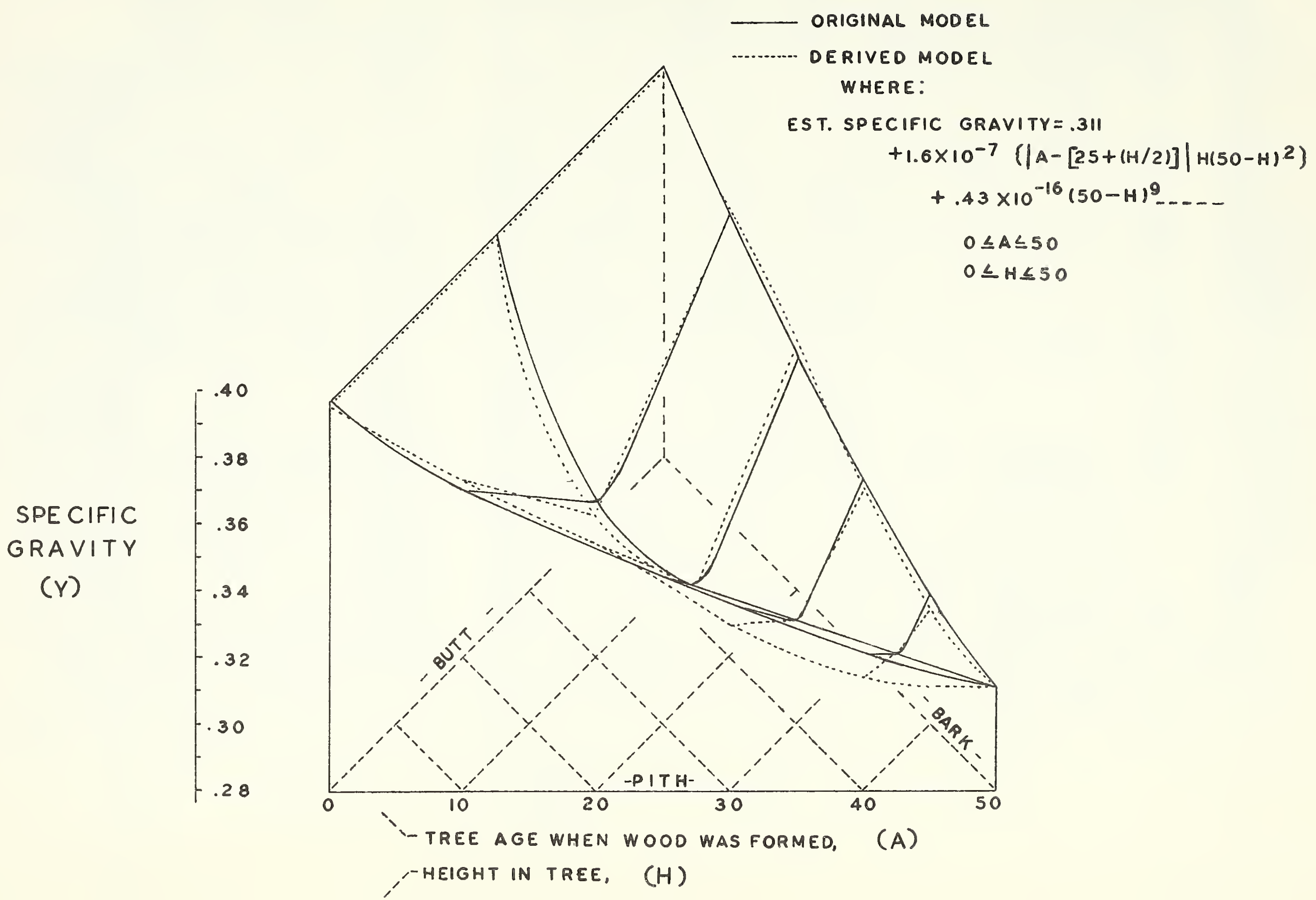
Example #2. The model that follows was developed from a very large number of observations. Feeling that this model is likely to be very representative of the population form, one might adopt it for testing in a new sample from a similar population.

Again, with the major objective of developing a reasonably accurate algebraic description, our computations[13/] follow about the same pattern as previously, so some of the detail of scaling is omitted. Instead, a brief verbal description of the individual algebraic steps and graphs showing regression components are included.

Note that "absolute value" is used to obtain increases in the same direction on either side of the curve of minimums (trough base) --- see "C". Also, scaling of a transformation may be based on slightly less or more than maximum values, so that an average coefficient may be used to represent two or more groups of representative points of the model. This is true in the scaling of both "C" and "G" in the computations that follow.

13/ Computations follow the model.

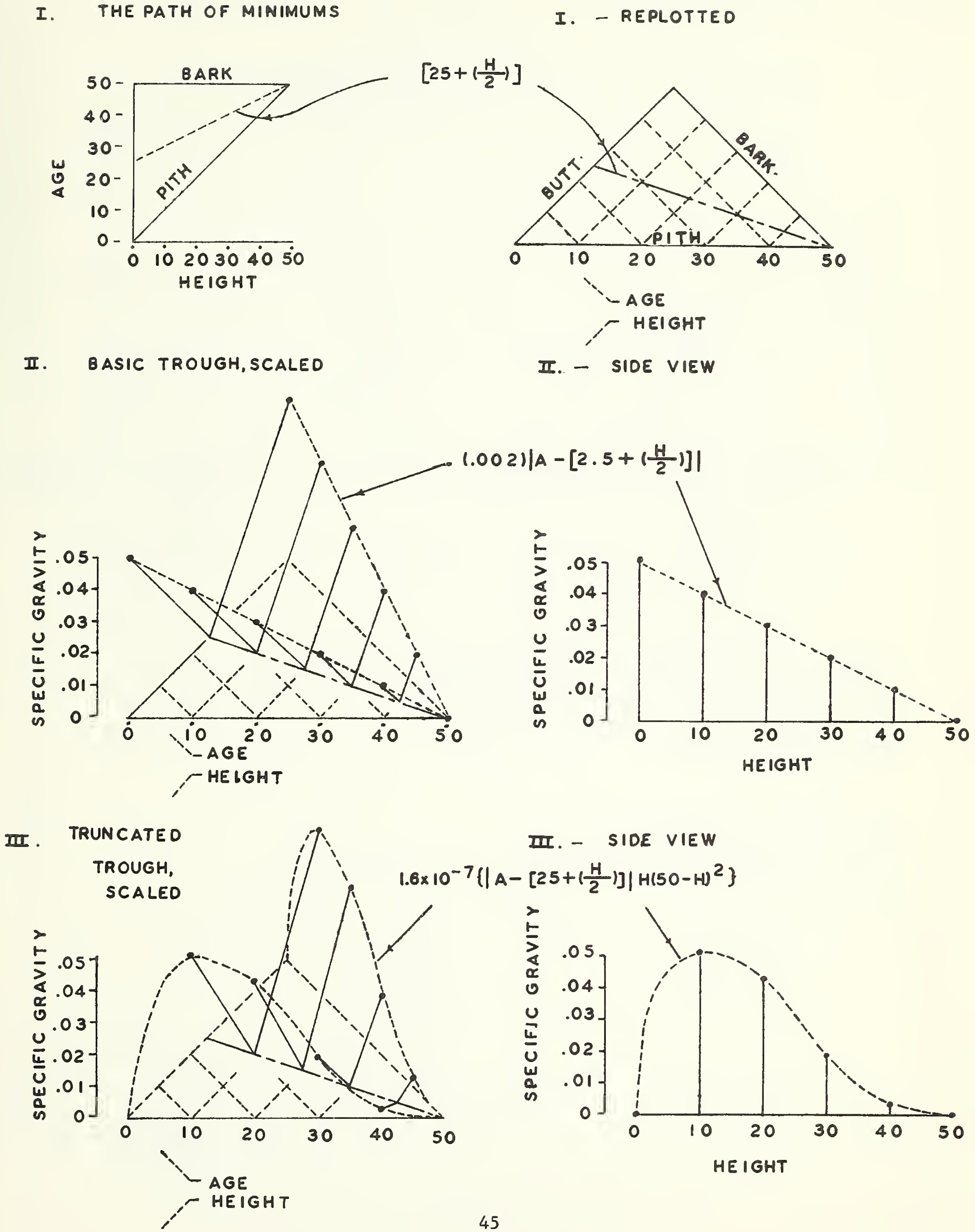
ORIGINAL - V.S. DERIVED MODEL



DEVELOPMENT OF $\hat{Y} = 1.6\text{x}10^{-7}\left[|A-[25+(H/2)]|H(50-H)^2\right] + .43\text{x}10^{-16}(50-H)^9 + .311$

H	A	Y	B	C		D	E	F	G	X_1	J	K	X_2	$\hat{Y}$
Model			25+(H/2)	|A-B|	Y_{LO}	$(Y-Y_{LO})$	.002(C)	$H(50-H)^2$	EF	$.8\text{x}10^{-4}$(G)	$(Y_{LO}-.311)(50-H)^9$	$.43\text{x}10^{-16}$(K)	$.311+X_1+X_2$
0	0	.396	25	25	.396	.000	.050	0	0	0	.085	$19.53\text{x}10^{14}$	.084	.395
	10	.396		15		.000	.030		0	0				.395
	20	.396		5		.000	.010		0	0				.395
	30	.396		5		.000	.010		0	0				.395
	40	.396		15		.000	.030		0	0				.395
	50	.396		25		.000	.050		0	0				.395
10	10	.370	30	20	.327	.043	.040	16M	640	.051	.016	$2.62\text{x}10^{14}$	.011	.373
	20	.348		10		.026	.020		320	.026				.348
	30	.327		0		.000	.000		0	.000				.322
	40	.345		10		.018	.020		320	.026				.348
	50	.373		20		.046	.040		640	.051				.373
20	20	.352	35	15	.312	.040	.030	18M	540	.043	.001	$.20\text{x}10^{14}$	.001	.355
	30	.325		5		.013	.010		180	.014				.326
	40	.321		5		.009	.010		180	.014				.326
	50	.350		15		.038	.030		540	.043				.355
30	30	.336	40	10	.311	.025	.020	12M	240	.019	.000	$.01\text{x}10^{14}$	.000	.330
	40	.311		0		.000	.000		0	.000				.311
	50	.334		10		.023	.020		240	.019				.330
40	40	.321	45	5	.311	.010	.010	4M	40	.003	.000	$.00\text{x}10^{14}$	.000	.314
	50	.319		5		.008	.010		40	.003				.314
50	50	.311	50	0	.311	.000	.000	0	0	.000	.000	$.00\text{x}10^{14}$	.000	.311
			B. Path of minimums	C. Basic trough transform	Smoothed low for trough	Response to be described	Basic trough transform scaled	To truncate butt of basic trough	Truncated trough transform	Scaled truncated trough transform	Elevation of trough above constant .311	Elevation transform	Scaled elevation transform	Compare with model "Y"

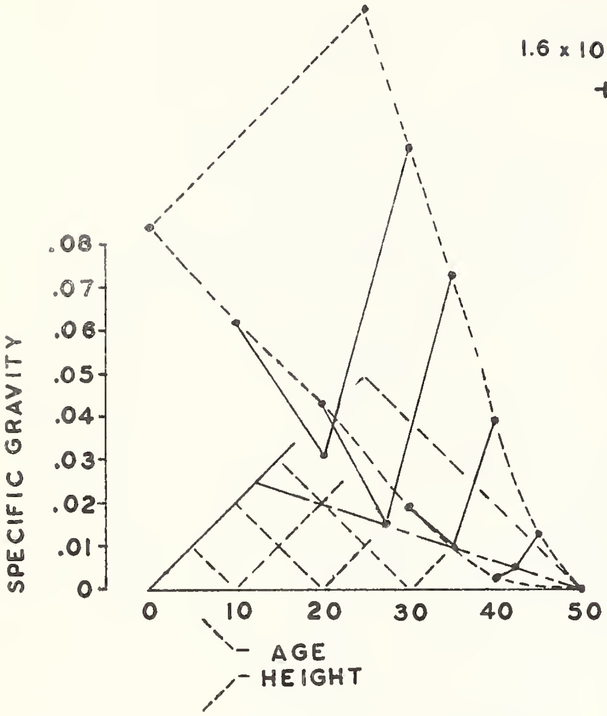
REGRESSION COMPONENTS

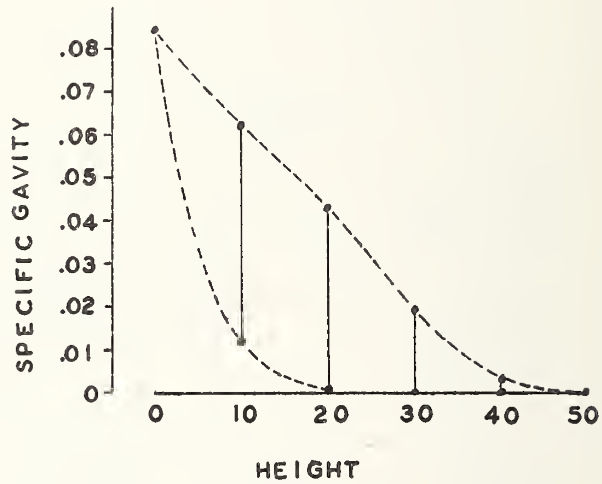


IV. ELEVATION OF TROUGH ABOVE CONSTANT, .311

SIDE VIEW

$$1.6 \times 10^{-7}\{|A-[25+(\tfrac{H}{2})]|H(50-H)^{2}\} + .43 \times 10^{-16}\ (50-H)^{9}$$





ADDING THE CONSTANT, .311, TO THE ABOVE FORM BRINGS THE DERIVED FORM REASONABLY CLOSE TO THAT OF THE ORIGINAL MODEL --- SIMPLY SHOWN IN THE SIDE VIEW BELOW:

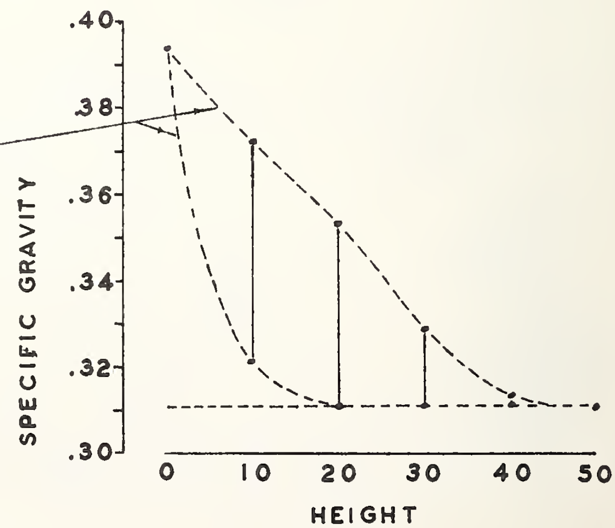
EST. SPECIFIC GRAVITY = .311

$$+1.6 \times 10^{-7}\{|A[25+(\tfrac{H}{2})]|H(50-H)^{2}\}$$

$$+.43 \times 10^{-16}(50-H)^{9}$$

$0 \leq A \leq 50$

$0 \leq H \leq 50$



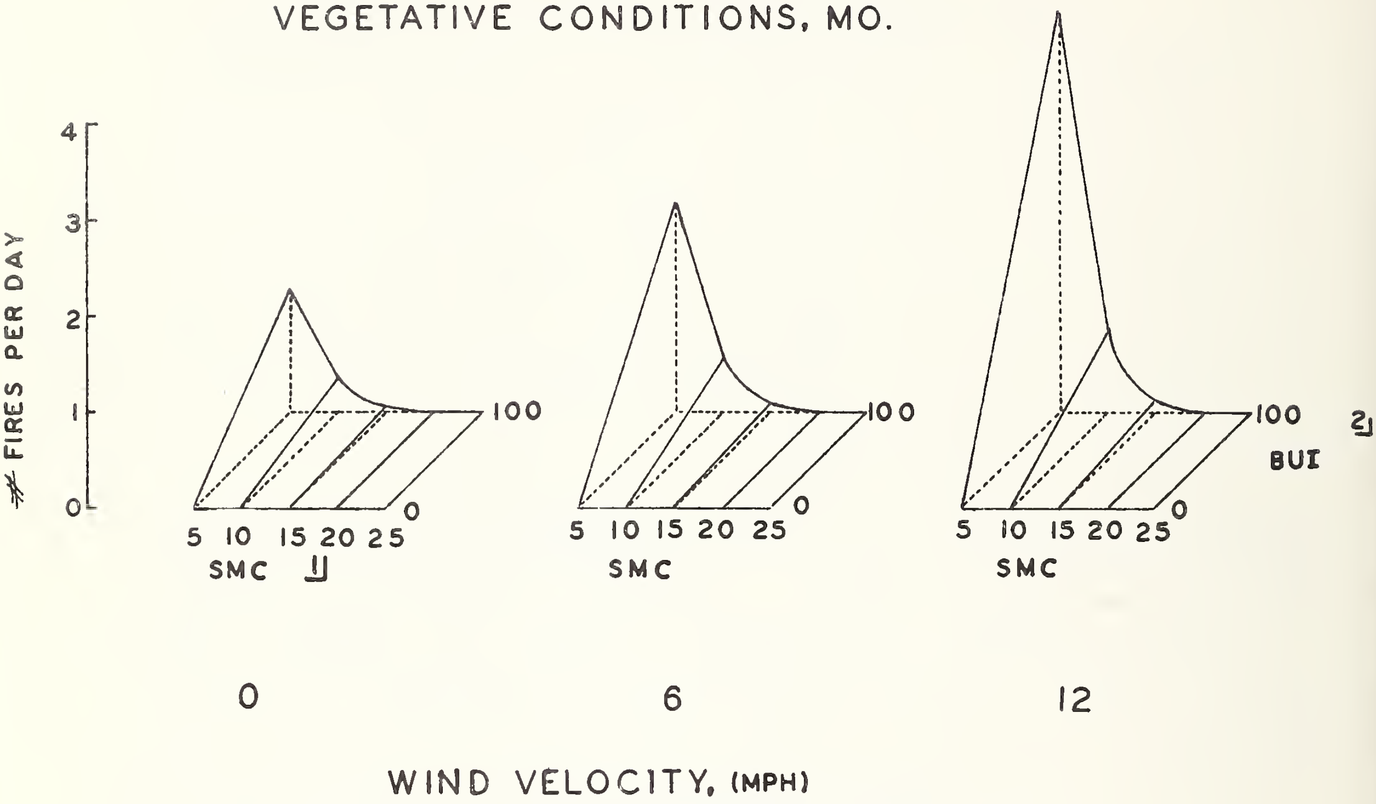
FOUR DIMENSIONS

A fairly simple case will be presented here to demonstrate application of the description system to representative points from three model surfaces.[14/] Note that each surface is a point in the fourth dimension. The continuity between surfaces will be provided by the term or terms in the algebraic description that involve the fourth variable. As before, take care that the terms selected do not move unrealistically between the representative points to which you scale.

The case at hand is based on a large amount of actual data from forest fire research. Surfaces were developed from data by procedures shown in the appendix. An additional refinement, that of smoothing number of fires for identical (SMC-BUI) points from each surface, was employed to arrive at adjusted responses for evenly distributed points in wind velocity. The three model surfaces below were the result.

14/ The same principles can be applied to five or more dimensional cases.

MODEL DEVELOPED GRAPHICALLY FROM DATA MEANS FOR GREEN VEGETATIVE CONDITIONS, MO.



1/ STICK MOISTURE CONTENT, (%)

2/ BUILD-UP INDEX

An algebraic description of this relation is:

$\# \text{ Fires} = .282\text{x}10^{-5}(BD^{3.1}) + .477\text{x}10^{-8}(A^{1.6}BD^{4.3})$---$5 \leq C \leq 20$

shown in the computations which follow.

The differences between representative points from this form and corresponding scaled points of the model, are not discernible at the scale shown.

DEVELOPMENT OF FORM FOR NUMBER FIRES PER DAY

A	B	C	Y_0	D			X_1	Y_1			E	X_2	$\hat{Y}$	CHECK
1/ WV	2/ BUI	3/ SMC	NUMBER FIRES (MODEL)	(20-C)	$D^{3.1}$	$BD^{3.1}$	$.282 \times 10^{-5}$ $(BD^{3.1})$ 4/	$Y_0 - X_1$	$D^{4.3}$	$A^{1.6}$	$(A^{1.6}BD^{4.3})$	$.477 \times 10^{-8}$ (E) 5/	$X_1 + X_2$	$\hat{Y} - Y_0$
0	0	5	0	15	4425	0	0	0	114,070	0	0	0	0	0
		10	0	10	1259	0	0	0	19,950		0	0	0	0
		20	0	0	0	0	0	0	0		0	0	0	0
	100	5	1.25			442,500	1.25	0			0	0	1.25	0
		10	.35	"	"	125,900	.36	- .01	"		0	0	.36	.01
		20	0			0	0	0			0	0	0	0
6	0	5	0			0	0	0		17.6	0	0	0	0
		10	0	"	"	0	0	0	"		0	0	0	0
		20	0			0	0	0			0	0	0	0
	100	5	2.15			442,500	1.25	.90			200,763,200	.96	2.21	.06
		10	.53	"	"	125,900	.36	.17	"		35,112,000	.17	.53	0
		20	0			0	0	0			0	0	0	0
12	0	5	0			0	0	0		53.3	0	0	0	0
		10	0	"	"	0	0	0	"		0	0	0	0
		20	0			0	0	0			0	0	0	0
	100	5	4.20			442,500	1.25	2.95			607,993,100	2.90	4.15	-.05
		10	.83	"	"	125,900	.36	.47	"		106,333,500	.51	.87	.04
		20	0			0	0	0			0	0	0	0

And, ~~Number Fires~~ Per Day = $X_1 + X_2 = .282 \times 10^{-5}(BD^{3.1}) + .477 \times 10^{-6}(A^{1.6}BD^{4.3})$, $5 \leq C \leq 20$

1/ Wind Velocity, (mph)
2/ Build-up index
3/ Stick moisture content (%)
4/ $1.25/442,500 = .282 \times 10^{-5}$
5/ $.477 \times 10^{-8}$ is slightly higher than the average of the two ratios, (.90/200,763,200) and (2.95/607,993,100)---minimizes differences from model.

NOTES ON THE FOREGOING FORM DEVELOPMENT:

1. $D^{3.1}$---selected because ($D^{3.1}$ at C = 5) vs. ($D^{3.1}$ at C = 10) has about the same ratio or slope, as Y_0 at (A = 0, B = 100, C = 5) vs. (A = 0, B = 100, C = 10.) --- e.g.

C	$D^{3.1}$	ratio	Y_0	ratio
5	4425	3.5	1.25	3.6
10	1259		.35	

When 4425 is scaled to 1.25---(1.25/4425) 4425 = ($.282\text{x}10^{-3}$) 4425 = 1.25, $D^{3.1}$ for C = 10, 1259, must also scale close to the value desired, .35---($.282\text{x}10^{-3}$) 1259 = .36

2. $D^{3.1}$ is multiplied by B in forming X_1 to:

   (a) suppress $D^{3.1}$ at B = 0

   (b) maintain linear additions with a change in B.

3. Y_1 shows the amount of Y_0 yet to be added in the algebraic description of the model.

4. The ratio between Y_1 for C = 5 and C = 10 is .90/.17 = 5.3 for A = 6 and 2.95/.47 = 6.3 for A = 12. It was decided here to adopt an average ratio for D^n of 5.7, provided by $D^{4.3}$ at C = 5 and 10---(114,070/19,950) = 5.7

   Then, $D^{4.3}$ scaled to .90, will yield close to the desired value, .17, for C = 10 at A = 6---and scaled to 2.95, will yield values close to the desired value, .47, for C = 10 at A = 12.

5. The ratio between Y_1 for C = 5, B = 100, A = 6 and 12 is (2.95/.90 = 3.3) and at C = 10 is (.47/.17 = 2.8). An aver-

age ratio of 3.0 was adopted and is provided by $A^{1.6}$ --- (53.3/17.6 = 3.0).

Scaling $A^{1.6}$ at (A = 12, C = 5) will then yield close to the desired value, .90 --- and similarly to .47 and .17, at (A = 6, C = 5).

6. The product $(A^{1.6}BD^{4.3})$ then yields a transformation that suppresses scaled values to zero at the appropriate points and allows reasonable additional curvature for both A and D. This composite quantity was then scaled to 2.90 instead of 2.95 in order to distribute some of the differences from Y_1 more evenly --- $(2.90/607{,}993{,}100)E = (.477\text{x}10^{-8})E$.

7. The happenstance presence of rather uniform slope changes in A = 6 and 12 permitted the ratio averaging technique and resulted in a brief, but fairly accurate description of the model. A more exacting description could probably be compiled with the use of separate, algebraic additions for each change in trend encountered.

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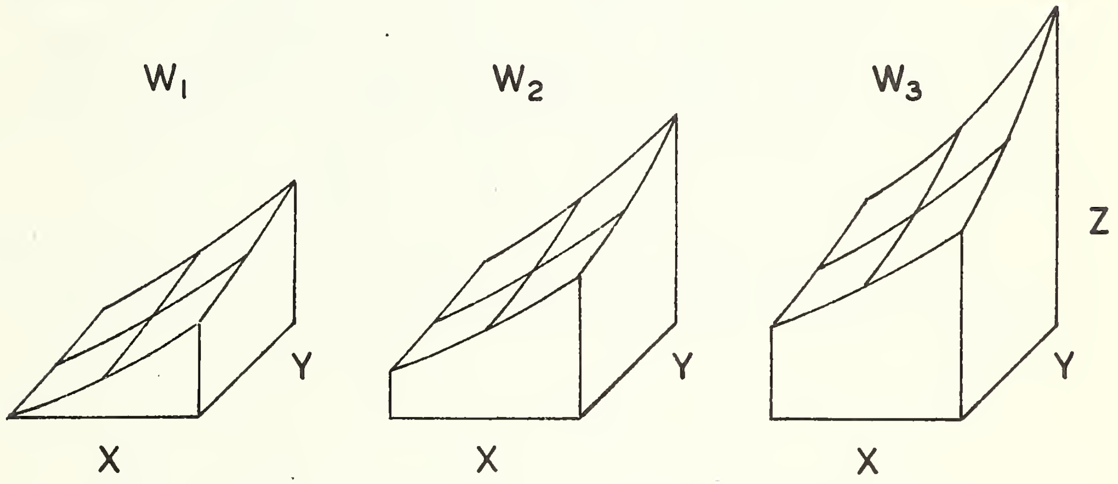
APPENDIX

GRAPHIC DEVELOPMENT OF SURFACES

Any relation involving two independent variables can be shown as a surface---more than two, as a series of surfaces---each at specified values of the remaining variables.

For example, a relation involving the dependent variable Z and the independent variables W, X, and Y might be presented as below.



Then it is only necessary to understand the derivation of a graph for a single surface in order to make the above presentation. Such a derivation scheme follows for cases where there is a reasonably wide dispersion of observations over the range of the independent variables. Without such dispersion, other techniques are more appropriate.

Given: Observations of Z on W, X and Y

Procedure:

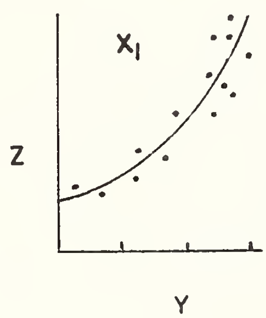
(a) Rate the independent variables as to suspected degree of importance in their effects on Z.

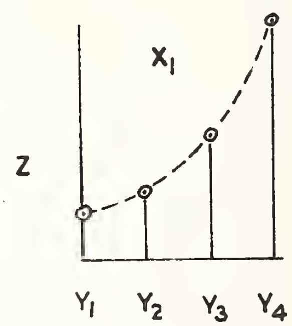
Separate the range of the least important variable, say "W", into four parts --- W_1 thru W_4 --- such that all observations are about equally distributed among them.

(b) Treat each W-group as follows:

Separate the range of the next most important variable, say "X", into four parts --- X_1 thru X_4 --- such that the observations are about equally distributed among them --- find the average X-value for each.

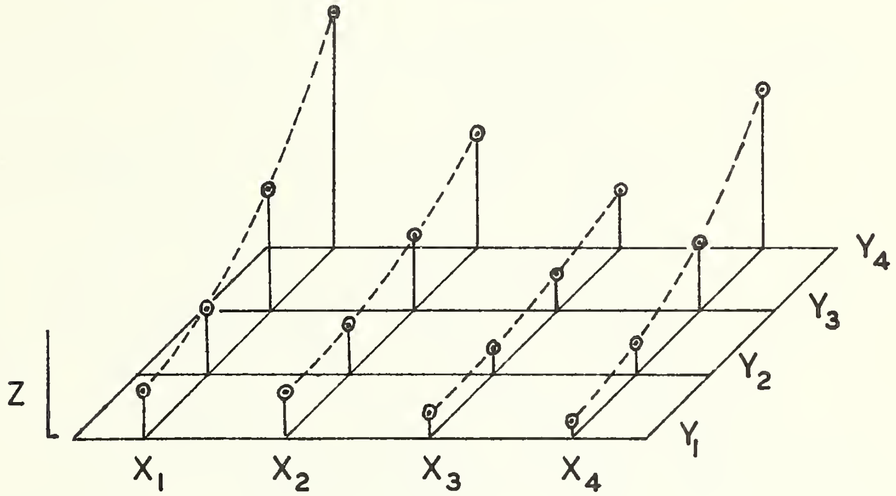
(c) For each X-group, fit a line through the Z-values plotted over corresponding Y-values, being careful to adhere to expected trends.



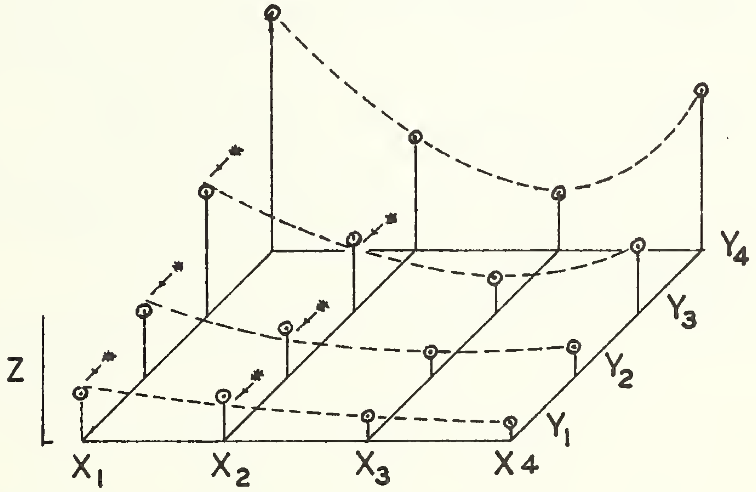


Repeat for remaining X-groups.

(d) Transfer these curves to a 3-dimensional graph. Be sure to plot each curve over the proper average X-value obtained in (b). Plot the 3rd dimension at a 45° angle to the 1st dimension.

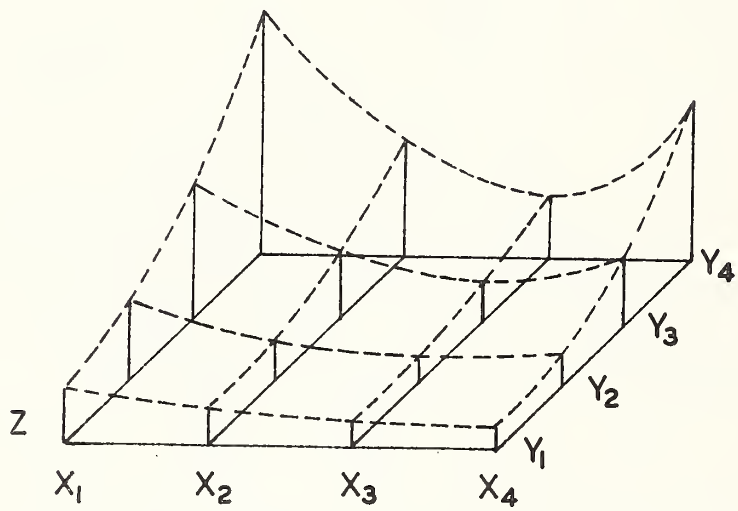


(e) Cross-curve lightly, as below, balancing + and - deviations as nearly as is practical.

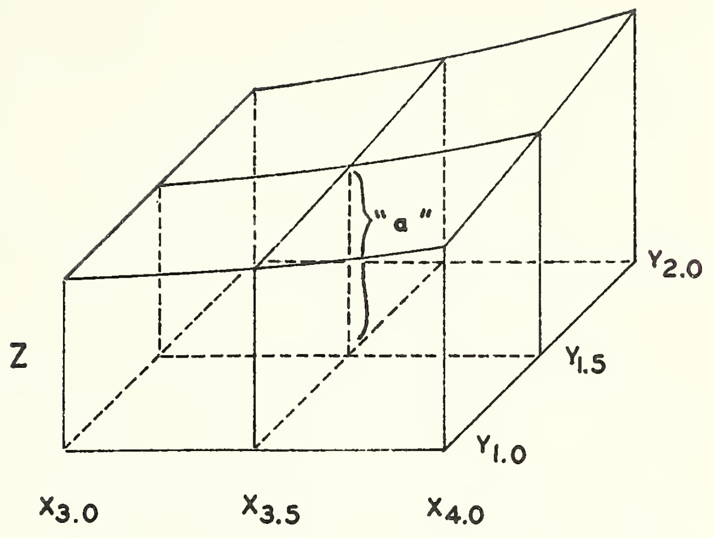


* = SMOOTHED Z – VALUES

(f) Using the curved Z-values above, fit a final smooth curve through the four points of each X-value to complete the surface, as below:



(g) However, it may be desirable to obtain surface Z-values for XY-values other than those represented (or, to interpolate). It is then necessary to locate the desired intercepts on existing curves and to join these intercepts with harmonizing curves. The surface value for the new combination of X-Y might then appear as "a" in the graph below.



(h) A further refinement can be applied by repeating (a) through (g) switching X and Y wherever either or both appear in the directions. You then have derived two surfaces. It but remains to obtain values from each surface for each of say, sixteen identical, XY-values --- and to average each pair of corresponding readings. The average surface may then be plotted from these new points.

The foregoing, with or without averaging (as time permits), is then repeated for W_{2}, 3, and 4.

Final surface values may then be obtained for all 64 WXY combinations and a formula derived for the graphic relation established.